Exercise 1. (Non-density of smooth functions in $W^{1,\infty}$) Let $I =]-1,1[$ and $f : I \to \mathbb{R}, x \mapsto |x|$. Show that $f \in W^{1,1}(I)$ and conclude that the closure of $C^{\infty}(I) \cap W^{1,\infty}(I)$ is strictly contained in $W^{1,\infty}(I)$.

Hint: continuous functions that converge in L^{∞} converge uniformly.

Exercise 2. (Unbounded Sobolev functions) Let $1 \leq p < \infty$ and $f \in C^{\infty}(\mathbb{R}_+^*, \mathbb{R})$. Let $R > 0$ $u : B(0,R) \subset \mathbb{R}^d, x \mapsto f(|x|)$.

1. Show that $u \in L^p(B_R(0))$ if and only if
$$\int_0^R r^{d-1}|f(r)|^p dr < \infty.$$
2. Assume that
$$\lim_{r \to 0} r^{d-1} f(r) = 0.$$
Show that $u \in W^{1,p}(B(0,R))$ if and only if $u \in L^p(B(0,R))$ and
$$\int_0^R r^{d-1}|f'(r)|^p dr < \infty.$$
Hint: One needs to show that the weak derivative is given by the pointwise derivative. To this end, split the domain into $B(0,\varepsilon)$ and the remainder and prove that the contribution coming from the small ball in the integration by parts formula is negligible.
3. Discuss the above results for $f(r) = r^{\gamma}$ with $\gamma \in \mathbb{R}$. More precisely, when do we have $u \in L^p(B(0,R))$ and when do we have $u \in W^{1,p}(B(0,R))$.

Exercise 3. (Sobolev function in dimension 1) Let $0 < a < b < \infty$, $I =]a,b[$. Show that $u \in W^{1,p}(I)$ if and only if there exists $v \in L^p(I)$ such that
$$u(x) = u(a) + \int_a^x v(t)dt.$$
Hint: Use Fubini's theorem to show that v has to be the weak derivative. For the converse implication, use the lemma of du Bois-Reymond.

Exercise 4. (Poincaré-type inequalities) Let $\Omega \subset \mathbb{R}^d$ be an open, bounded set with Lipschitz boundary and $1 \leq p \leq \infty$. Let $S \subset W^{1,p}(\Omega)$ be a closed subspace such that the only function satisfying $\nabla u = 0$ is the zero function. Show that there exists a universal constant $C > 0$ such that
$$\|u\|_{\mathrm{L}^p(\Omega)} \leq C \|\nabla u\|_{\mathrm{L}^p(\Omega)} \qquad \text{for all } u \in S.$$
Hint: Argue by contradiction and consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subset S$ such that $\|u_n\|_{\mathrm{L}^p(\Omega)} \geq n \|\nabla u_n\|_{\mathrm{L}^p(\Omega)}$. After an appropriate rescaling, use the Rellich-Kondrachov compactness theorem.